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Symmetries and differential equations

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Abstract. The knowledge of the maximal Lie group or abstract monoid of symmetries of an ordinary non-singular differential equation (or system of equations) allows us to obtain solutions of them. Traditional similarity analysis of point transformations is extended to non-point transformations (inclusion of derivatives), giving analytic expressions for solutions, where previously only numerical methods were used. Examples are given, and the didactic aspect is emphasised.

1. Introduction

It has been known since Lie (1888–1927) that the theory of continuous transformation groups gives promise of providing a canonical method to deal with complicated systems of ordinary or partial differential equations. The method, known also as similarity analysis, is by no means new; in fact, the fundamental ideas date back to the end of the last century and are the product of one man: the Swedish mathematician Sophus Lie.

Later development and application of similarity analysis can be found in the works of Cohen (1931), Birkhoff (1950), Michal (1952), Morgan (1952), Ovsjannikov (1962), Na and Hansen (1971) and Bluman and Cole (1974). The inclusion of contact transformations as symmetries of a differential equation with the aim of constructing new solutions from known ones, or reducing the number of variables of a partial differential equation (reduction of order in an ordinary differential equation), did not bring about an essential generalisation of traditional similarity analysis of point transformations, and only recent work by Anderson *et al* (1972; Anderson and Davison 1974) and Gonzalez-Gascon (1977) produced new hopes for this elegant mathematical theory. Our aim in this article is, besides presenting traditional similarity analysis in a didactical form, a generalisation to non-point transformations and a presentation of a method for obtaining non-point symmetries of differential equations. We deal only with non-singular ordinary differential equations (or systems of ordinary differential equations) as the extension to singular ordinary differential equations or systems is straightforward. The problem of initial or boundary values and of partial differential equations will be treated in subsequent papers.

Illustrative examples showing the powers of this method are given.

Finally, we would consider our task accomplished if we could convince the scientific community that, before going to the computer to solve a differential equation by numerical methods, one should try via generalised similarity analysis to obtain an analytic expression for the solution or to construct new solutions from known ones, or at least to reduce the number of variables or the order of the differential equation.

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2. First-order non-singular ordinary differential equation

Let us consider the general expression for the first-order non-singular ordinary differential equation:

$$dy/dx - f(x, y) = 0 \tag{2.1}$$

and let us seek the *symmetries* of this equation; i.e. the set of transformations of the dependent and independent variables under which equation (2.1) is 'conformally invariant'.

Set

$$\bar{y} = \bar{y}(x, y), \qquad \bar{x} = \bar{x}(x, y), \tag{2.2}$$

and the new differential equation will be 'conformally invariant' to (2.1), if

$$d\bar{y}/d\bar{x} - f(\bar{x}, \bar{y}) = G(x, y)[dy/dx - f(x, y)] = 0 \tag{2.3}$$

where $G(x, y)$, the conformal factor, is different from zero in a region of the phase space (x, y).

An equivalent definition of symmetry of a differential equation is any transformation that, when applied to a solution of a differential equation, gives us a new solution of the differential equation. This important property of the symmetry of a differential equation tells us that the knowledge of such symmetries will help us to obtain new solutions from known ones.

There are many types of symmetries. For example:
discrete symmetries:

$$\frac{dy}{dx} = y \text{ is symmetric under } \begin{array}{l} y \to \bar{y} = -y, \\ x \to \bar{x} = x; \end{array}$$

periodic symmetries:

$$\frac{dy}{dx} = F(x)y \text{ (where } F(x) = F(x+T)\text{) is symmetric under } \begin{array}{l} x \to \bar{x} = x + T, \\ y \to \bar{y} = y; \end{array}$$

continuous symmetries:

$$\frac{dy}{dx} = y \text{ is symmetric under } \begin{array}{l} y \to \bar{y} = ay, \\ x \to \bar{x} = x; \end{array}$$

and we can think also of more complicated situations, where (2.2) describes a functional or quasidifferential operator acting on the dependent and independent variables.

If we know *all* the symmetries of a differential equation, we will say that we have the *maximal set of symmetries*, and we conjecture that most of the differential equations dealt with science and which represent natural phenomena have symmetries. This is a fundamental hypothesis of our work†. Whereas it is clear that a differential equation defines a maximal set of symmetries, we next conjecture that a maximal set of symmetries uniquely determines a differential equation. However, in the present article we will confine our investigation to very special types of symmetries, and only in

† Perhaps it might be possible to construct some very special differential equation, where it can be shown that it has no symmetry (besides that of the identity transformation), but the author doubts if this equation would have any scientific use or application apart from pure academic interest. Certainly nothing is known in this direction; the problem, as we will see, is to find the symmetries, which sometimes can be a very hard job.

subsequent articles will we search for the maximum set of symmetries. This conjecture could be helpful in elementary particle physics, where we have some groups and discrete symmetries that allow us to classify elementary particles. But we know very little about the dynamics of these particles, including whether or not it can be described by a differential or integrodifferential equation.

We now return to our purpose of finding symmetries of ordinary differential equations, and we define a *point-transformation symmetry*†

$$x \to \bar{x} = \bar{x}(x, y), \qquad y \to \bar{y} = \bar{y}(x, y), \tag{2.4}$$

where equation (2.3) holds.

A symmetry will be callled *local differential* (non-point transformation) if the prescription used to transform a solution of a differential equation into another solution is of the form

$$x \to \bar{x} = \bar{x}\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}, \ldots\right), \qquad y \to \bar{y} = \bar{y}\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}, \ldots\right) \tag{2.5}$$

(a finite or infinite number of derivatives can be included). We again take the example

$$\mathrm{d}y/\mathrm{d}x - y = 0 \tag{2.6}$$

and try to find the point transformation symmetries (2.4) of this equation‡. In other words, we want to find a point transformation (2.4) such that, when it is applied to our equation, we obtain a new equation differing from the former by just a conformal factor:

$$\mathrm{d}\bar{y}/\mathrm{d}\bar{x} - \bar{y} = G(x, y)(\mathrm{d}y/\mathrm{d}x - y) = 0. \tag{2.7}$$

So, if (2.6) is satisfied for a known solution, (2.4) applied to it gives us a new solution of (2.6).

Now we can write the left-hand side of (2.7) as

$$\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} - \bar{y} = \frac{(\partial\bar{y}/\partial x)\,\mathrm{d}x + (\partial\bar{y}/\partial y)\,\mathrm{d}y}{(\partial\bar{x}/\partial x)\,\mathrm{d}x + (\partial\bar{x}/\partial y)\,\mathrm{d}y} - \bar{y}(x, y),$$

and to cast it in the form (2.7) we can assume

$$\text{(i)} \quad \bar{y} = \bar{y}(y), \qquad \text{(ii)} \quad \bar{x} = \bar{x}(x).$$

This implies

$$\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} - \bar{y} = \frac{(\mathrm{d}\bar{y}/\mathrm{d}y)\,\mathrm{d}y}{(\mathrm{d}\bar{x}/\mathrm{d}x)\,\mathrm{d}x} - \bar{y}(y).$$

We define

$$\bar{y}(y) := G(y)y$$

and assume

$$\text{(iii)} \quad \frac{(\mathrm{d}G/\mathrm{d}y)y + G(y)}{(\mathrm{d}\bar{x}/\mathrm{d}x)} = G(y).$$

† The name comes from the fact that the transformation relates two different points of phase space (x, y) and $(\bar{x}, \bar{y})$.

‡ For first-order ordinary differential equations it makes no sense to consider local differential symmetries, because the differential equation can be introduced in the right-hand side of (2.5), resulting in a point transformation symmetry.

From (iii) we conclude that $\mathrm{d}\bar{x}/\mathrm{d}x$ must be equal to a constant,

$$\mathrm{d}\bar{x}/\mathrm{d}x = a \Rightarrow \bar{x} = ax + b,$$

and we have to solve the following equation to obtain the conformal factor:

$$y\,\mathrm{d}G/\mathrm{d}y + (1-a)G(y) = 0. \tag{2.8}$$

The general solution is given by

$$G(y) = cy^{a-1}.$$

We now know that the point transformation

$$x \to \bar{x} = ax + b, \qquad y \to \bar{y} = cy^{a}, \qquad a, b, c \text{ are real constants,}$$

is a symmetry of equation (2.6); i.e. equation (2.7) holds.

We can immediately raise questions about this method of finding the symmetries of equation (2.6). First, it could be that our three assumptions are too restrictive and that there are more symmetries that we did not obtain. Second, for more complicated equations of first or higher order, we will not be able to make such straightforward assumptions as we did. Finally, the equations for the conformal factor G could be more difficult than the original equation. At this point the fundamental ideas of Lie (1888–1927) will come to our help to settle the problem.

Let us suppose that the point transformation (2.4) is a one-parameter transformation:

$$\bar{x} = \bar{x}(x, y; \alpha), \qquad \bar{y} = \bar{y}(x, y; \alpha), \tag{2.9}$$

and we assume the analytic dependence of (2.9) on α in the neighbourhood of the identity transformation, defined as

$$\bar{x}(x, y; \alpha_0) := x, \qquad \bar{y}(x, y; \alpha_0) := y. \tag{2.10}$$

Setting $\alpha := \alpha_0 + \varepsilon$ and expanding in powers of ε, we obtain

$$\bar{x}(x, y; \alpha) = \bar{x}(x, y; \alpha_0 + \varepsilon) = x + \varepsilon \frac{\partial \bar{x}}{\partial \alpha}\bigg|_{\alpha=\alpha_0} + \frac{\varepsilon^2}{2!}\frac{\partial^2 \bar{x}}{\partial \alpha^2}\bigg|_{\alpha=\alpha_0} + \ldots,$$

$$\bar{y}(x, y; \alpha) = \bar{y}(x, y; \alpha_0 + \varepsilon) = y + \varepsilon \frac{\partial \bar{y}}{\partial \alpha}\bigg|_{\alpha=\alpha_0} + \frac{\varepsilon^2}{2!}\frac{\partial^2 \bar{y}}{\partial \alpha^2}\bigg|_{\alpha=\alpha_0} + \ldots.$$

Let us define

$$\frac{\partial \bar{x}}{\partial \alpha}\bigg|_{\alpha=\alpha_0} := \zeta(x, y), \qquad \frac{\partial \bar{y}}{\partial \alpha}\bigg|_{\alpha=\alpha_0} := \eta(x, y), \tag{2.11}$$

so that

$$\frac{\partial^2 \bar{x}}{\partial \alpha^2}\bigg|_{\alpha=\alpha_0} = \frac{\partial \zeta}{\partial x}\frac{\partial \bar{x}}{\partial \alpha}\bigg|_{\alpha=\alpha_0} + \frac{\partial \zeta}{\partial y}\frac{\partial \bar{y}}{\partial \alpha}\bigg|_{\alpha=\alpha_0} = \zeta\frac{\partial \zeta}{\partial x} + \eta\frac{\partial \zeta}{\partial y},$$

$$\frac{\partial^2 \bar{y}}{\partial \alpha^2}\bigg|_{\alpha=\alpha_0} = \frac{\partial \eta}{\partial x}\frac{\partial \bar{x}}{\partial \alpha}\bigg|_{\alpha=\alpha_0} + \frac{\partial \eta}{\partial y}\frac{\partial \bar{y}}{\partial \alpha}\bigg|_{\alpha=\alpha_0} = \zeta\frac{\partial \eta}{\partial x} + \eta\frac{\partial \eta}{\partial y},$$

etc, so we can write the Taylor expansion as

$$\bar{x} = x + \varepsilon\zeta(x, y) + \frac{\varepsilon^2}{2!}\left(\zeta\frac{\partial\zeta}{\partial x} + \eta\frac{\partial\zeta}{\partial y}\right) + \ldots,$$
$$\bar{y} = y + \varepsilon\eta(x, y) + \frac{\varepsilon^2}{2!}\left(\zeta\frac{\partial\eta}{\partial x} + \eta\frac{\partial\eta}{\partial y}\right) + \ldots. \qquad (2.12)$$

This power series will in general converge in some neighbourhood of $\varepsilon = 0$ and represent the global transformation (2.9).

Let us consider ε so small that higher orders of ε can be neglected†; we return to equation (2.1) with the intention of casting it into the form (2.3) after using (2.12). From (2.3)

$$\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} - f(\bar{x}, \bar{y}) = \frac{(\partial\bar{y}/\partial x)\,\mathrm{d}x + (\partial\bar{y}/\partial y)\,\mathrm{d}y}{(\partial\bar{x}/\partial x)\,\mathrm{d}x + (\partial\bar{x}/\partial y)\,\mathrm{d}y} - f(x + \varepsilon\zeta, y + \varepsilon\eta) = 0$$
$$= \frac{\mathrm{d}y + \varepsilon[(\partial\eta/\partial x)\,\mathrm{d}x + (\partial\eta/\partial y)\,\mathrm{d}y]}{\mathrm{d}x + \varepsilon[(\partial\zeta/\partial x)\,\mathrm{d}x + (\partial\zeta/\partial y)\,\mathrm{d}y]} - f(x, y) - \varepsilon\left(\zeta\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y}\right)$$
$$= \left[\frac{\mathrm{d}y}{\mathrm{d}x} + \varepsilon\left(\frac{\partial\eta}{\partial x} + \frac{\partial\eta}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}\right)\right]\left[1 - \varepsilon\left(\frac{\partial\zeta}{\partial x} + \frac{\partial\zeta}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}\right)\right] - f(x, y) - \varepsilon\left(\zeta\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y}\right) = 0.$$

With this

$$\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} - f(\bar{x}, \bar{y}) = \frac{\mathrm{d}y}{\mathrm{d}x} - f(x, y) + \varepsilon\left[\frac{\partial\eta}{\partial x} + \left(\frac{\partial\eta}{\partial y} - \frac{\partial\zeta}{\partial x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{\partial\zeta}{\partial y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \zeta\frac{\partial f}{\partial x} - \eta\frac{\partial f}{\partial y}\right] = 0. \qquad (2.13)$$

The last expression can be cast into the desired form, if we assume that the conformal factor $G(x, y)$ can be Taylor expanded around the value α_0 of the parameter, and define

$$G(x, y; \alpha_0) := 1,$$

so that, to first orders in ε,

$$G(x, y; \alpha) = 1 + \varepsilon\,\partial G/\partial\alpha|_{\alpha=\alpha_0}$$

and by identification

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x} - f(x, y)\right)\frac{\partial G}{\partial\alpha}\bigg|_{\alpha=\alpha_0} = \frac{\partial\eta}{\partial x} + \left(\frac{\partial\eta}{\partial y} - \frac{\partial\zeta}{\partial x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{\partial\zeta}{\partial y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \zeta\frac{\partial f}{\partial x} - \eta\frac{\partial f}{\partial y}.$$

We can now state the following theorem.

Theorem (Bluman and Cole 1974). $\bar{x} = x + \varepsilon\zeta(x, y)$ and $\bar{y} = y + \varepsilon\eta(x, y)$ define uniquely (by (2.12)) a symmetry transformation for equation (2.1), if and only if

$$\frac{\partial\eta}{\partial x} + \left(\frac{\partial\eta}{\partial y} - \frac{\partial\zeta}{\partial x}\right)f - \frac{\partial\zeta}{\partial y}f^2 - \zeta\frac{\partial f}{\partial x} - \eta\frac{\partial f}{\partial y} = 0. \qquad (2.14)$$

Some conclusions from our previous calculations can be stated.

(a) If the infinitesimal transformation is given:

$$x \to \bar{x} = x + \varepsilon\zeta(x, y), \qquad y \to \bar{y} = y + \varepsilon\eta(x, y), \qquad (2.15)$$

† Lie (1888–1927) showed that there is no loss of generality under this assumption, and he called $\bar{x} = x + \varepsilon\zeta(x, y)$, $\bar{y} = y + \varepsilon\eta(x, y)$, the infinitesimal transformation,

then the infinitesimal transformation for the first derivative, also called the *first extension*, is

$$\frac{dy}{dx} \to \frac{d\bar{y}}{d\bar{x}} = \frac{dy}{dx} + \varepsilon\left[\frac{\partial \eta}{\partial x} + \left(\frac{\partial \eta}{\partial y} - \frac{\partial \zeta}{\partial x}\right)\frac{dy}{dx} - \frac{\partial \zeta}{\partial y}\left(\frac{dy}{dx}\right)^2\right]. \tag{2.16}$$

(b) From (2.15) there are two methods of constructing and defining uniquely the finite transformation; one, as already stated in the theorem, is from (2.12), and the other is through the definition (2.11); i.e. we must solve what is called in the literature the *characteristic equation*:

$$dx/\zeta(x, y) = dy/\eta(x, y) = d\alpha. \tag{2.17}$$

Example.

$$x \to \bar{x} = x + \varepsilon x, \qquad y \to \bar{y} = y + \varepsilon \sin y.$$

Then $dx/x = d\alpha$ and $dy/\sin y = d\alpha$, implying

$$\ln x = \alpha + Q, \qquad \ln \tan y = \alpha + P,$$

where Q and P are integration functions such that when the parameter α is zero we obtain the identity transformation; i.e.

$$\ln x = \alpha + \ln \bar{x}, \qquad \ln \tan y = \alpha + \ln \tan \bar{y},$$

so that the finite transformation is

$$x \to \bar{x} = e^{-\alpha} x, \qquad y \to \bar{y} = \tan^{-1}(e^{-\alpha} \tan y).$$

Note that if the characteristic equation (2.17) cannot be solved for known functions ζ and η, then we can always return to (2.12) and content ourselves with the fact that, if we know a solution of our differential equation, then the new† one will be expressed in terms of a power series expansion.

Example. The Bernoulli equation

$$\frac{dy}{dx} = \frac{1 + x \sin x}{x^3} y^3 - (\sin x) y$$

has a particular solution $y = x$. It can also be shown by (2.14) that it is conformally invariant under the infinitesimal transformation

$$x \to \bar{x} = x, \qquad y \to \bar{y} = y + \varepsilon y^3 e^{-2\cos x},$$

and the characteristic equations are

$$dx/0 = dy/y^3 e^{-2\cos x} = d\sigma.$$

Integrating, we obtain the following finite transformations:

$$x \to \bar{x} = x, \qquad y \to \bar{y} = y/(1 - 2\sigma y^2 e^{-2\cos x})^{1/2},$$

and the new family of solutions is

$$y_{\text{New}} = x/(1 - 2\sigma x^2 e^{-2\cos x})^{1/2}.$$

† It is possible that we will obtain the same solution again.

(c) From equation (2.14) we conclude that there is an infinite class of infinitesimal transformations which leave first-order ordinary differential equations conformally invariant, because we have two unknowns ζ, η for one partial first-order differential equation. This answers some questions. First, for the equation $\mathrm{d}y/\mathrm{d}x - y = 0$ we will have more symmetries than those obtained by the method of assumptions. Second, there is no general solution for equation (2.14), and we must develop *ad hoc* methods to solve it, as we will see later on.

(d) Eisenhart (1966) has proven that the infinitesimal transformations that leave a first-order ordinary differential equation conformally invariant describe a group. This fact is true also for differential equations of higher order (see Gonzalez-Gascon 1977), as long as we deal with point transformations; but considering local differential symmetries, we cannot guarantee the existence of an inverse transformation; so in general we will have only a monoid structure if we also include local differential symmetries for our equations.

(e) Once equation (2.14) has been solved for a first-order ordinary differential equation, then the integrating factor for this equation can be obtained directly from the knowledge of ζ and η (see Cohen 1931, Bluman and Cole 1974):

$$I = 1/(\eta - f\zeta). \tag{2.18}$$

If we know two different integrating factors $I_1 \neq I_2$, i.e. two pairs of solutions (ζ_1, η_1) and (ζ_2, η_2) of equation (2.14), then the general solution of our first-order ordinary differential equation can be obtained from Cohen (1931):

$$I_1/I_2 = \text{constant}.$$

3. Solving the fundamental equation for the infinitesimals

Equation (2.14)

$$\frac{\partial \eta}{\partial x} + \left(\frac{\partial \eta}{\partial y} - \frac{\partial \zeta}{\partial x}\right) f - \frac{\partial \zeta}{\partial y} f^2 - \zeta \frac{\partial f}{\partial x} - \eta \frac{\partial f}{\partial y} = 0$$

is the heart of the problem for first-order ordinary differential equations, because once η and ζ are known, we know the integrating factor or can construct new solutions from known ones. But we must be careful;

$$\zeta = \zeta(x, y), \qquad \eta = f\zeta(x, y), \tag{3.1}$$

is a solution of (2.14) and the integrating factor is not defined. Apart from the solution (3.1) and the trivial one, any other pair is suitable for our purposes.

As we have already stated, we cannot find the general solutions of (2.14) because the problem is overdetermined, but we can state that if (η_1, ζ_1) and (η_2, ζ_2) are solutions of (2.14), then $(\eta_1 + \eta_2, \zeta_1 + \zeta_2)$ is also a solution. That is a consequence of the linearity of this first-order partial differential equation. From that, we can conclude also that the first-order ordinary differential equation will have an *infinite parametric* group of symmetries, because it has an infinite number of monoparametric symmetries and because of the linearity property of the equation. Second, we can try to invert the problem, and, given η and ζ, to solve the quasilinear first-order partial differential equation for f via the method of characteristics.

Example. Let $\eta = 0$ and $\zeta = y$; then the quasilinear first-order partial differential equation for f is, from (2.14),

$$f^2 + y\,\partial f/\partial x = 0,$$

and solving, we obtain the general solution

$$f = y/(x + \psi(y)).$$

This means that the following ordinary first-order differential equation has the expected symmetry:

$$y' = y/(x + \psi(y)).$$

This fundamental idea, that, given symmetries *a priori*, we can obtain the differential equations, can be very helpful in elementary particle physics, where an equation describing the dynamics of the particles is needed and only symmetries for the particles are known. On the other hand, it allows us to construct tables of differential equations conformally invariant under a certain symmetry transformation (see Cohen 1931, Bluman and Cole 1974).

Now returning to equation (2.14), let us assume

(i) $\zeta = 0, \qquad \eta \neq 0.$

Then we obtain the following first-order quasilinear partial differential equation:

$$\partial\eta/\partial x + f\,\partial\eta/\partial y = \eta\,\partial f/\partial y,$$

which can be solved by the method of characteristics (Bluman and Cole 1974):

$$\frac{dx}{ds} = 1 \Rightarrow x = s + \text{constant}, \qquad \frac{dy}{ds} = f(x, y) \Rightarrow \frac{dy}{ds} = f(s, y),$$

$$d\eta/ds = \eta\,\partial f/\partial y.$$

We observe here that this ansatz doesn't help us, because one of the characteristic equations is the equation that we want to solve†.

(ii) $\zeta = 0, \qquad \eta = \eta(f, \partial f/\partial x, \partial f/\partial y).$

Introducing this in equation (2.14), a straightforward calculation gives

$$\eta = \exp\left(A + \int \frac{df}{(f_x/f_y) + f}\right), \qquad \zeta = 0,$$

where $f_x := \partial f/\partial x$ and $f_y := \partial f/\partial y$.

(iii) $\zeta = \zeta(f, \partial f/\partial x, \partial f/\partial y), \qquad \eta = 0.$

We obtain

$$\zeta = \exp\left(B - \int \frac{df}{(f_y/f_x)f^2 + f}\right), \qquad \eta = 0.$$

Condensing both cases (ii) and (iii) into one, because of the linearity of equation (2.14) we obtain as a particular solution of (2.14)

$$\eta = \exp\left(A + \int \frac{df}{(f_x/f_y) + f}\right), \qquad \zeta = \exp\left(B - \int \frac{df}{(f_y/f_x)f^2 + f}\right). \tag{3.2}$$

† We get the same if we make the ansatz $\zeta \neq 0$ and $\eta = 0$.

If f_x/f_y is an arbitrary function† of f, but not equal to $-f$, we can in principle integrate (3.2) and obtain ζ and η explicitly.

Example.

$$y' = (x+y)^n \Rightarrow f(x, y) = (x+y)^n.$$

Then

$$\frac{f_x}{f_y} = 1 \Rightarrow \eta = \exp\left(A + \int \frac{\mathrm{d}f}{f+1}\right) = \mathrm{e}^A[f+1] = \mathrm{e}^A[(x+y)^n + 1].$$

If we assume $\zeta \equiv 0$, the integrating factor by (2.18) is

$$I = \mathrm{e}^{-A}/[(x+y)^n + 1].$$

The fact that $f_x/f_y = \varphi(f)$ and $\varphi(f) \neq -f$ can be used to enlarge our table of integrable first-order differential equations.

Let us solve the equation

$$f_x - \varphi(f) f_y = 0 \tag{3.3}$$

by the method of characteristics; then

$$f = F(y + \varphi(f)x) \tag{3.4}$$

where F and φ are arbitrary functions, i.e. we just solve the algebraic problem of constructing the function f such that (3.3) is satisfied. Supposing that F has an inverse F^{-1}, then

$$F^{-1}(f) = y + \varphi(f)x, \tag{3.5}$$

and by definition let us call this

$$g := F^{-1}(\varphi).$$

Then from (3.5) we have

$$g = y + \varphi(f)x.$$

But $f = F(g)$. So

$$g = y + [\varphi \circ F](g)x$$

and $\varphi \circ F := \Psi$, so that the algebraic equation to be solved first for g, given an arbitrary Ψ, is

$$g = y + \Psi(g)x.$$

Once g is known we apply another arbitrary function F to it and obtain an f.

Example. Let Ψ be the identity function, so we have the algebraic equation

$$g = y + gx \Rightarrow g = y/(1-x).$$

Now let us again take for F the identity function; we obtain $f = y/(1-x)$ and the ordinary differential equation is

$$y' = y/(1-x) \Rightarrow f_x/f_y = y/(1-x) = f.$$

† f_x/f_y could also be a constant.

So the integrating factor is

$$I = \mathrm{e}^{-A} \Big/ \exp\left(\int \frac{\mathrm{d}f}{2f}\right) = \mathrm{e}^{-A}\left(\frac{1-x}{y}\right)^{1/2}.$$

(iv) Before we finish with this section let us mention another way to solve (2.14). If $f(x, y)$ is a simple polynomial in x and y, it is worthwhile assuming $\zeta = 0$ and η also a polynomial in x and y, so that we can determine it through (2.14), identifying coefficients.

Now we will attack the problem of second-order, non-singular ordinary differential equations, and through this we will find yet another way of solving equation (2.14).

4. The second-order non-singular ordinary differential equation

$$\mathrm{d}^2y/\mathrm{d}x^2 = f(x, y, y'). \tag{4.1}$$

The treatment of the second- or higher-order non-singular ordinary differential equation is essentially along the same lines as that of the first-order non-singular ordinary differential equation. We can consider local differential symmetries for this equation:

$$\begin{aligned} x \to \bar{x} &= \bar{x}(x, y, \mathrm{d}y/\mathrm{d}x) = x + \varepsilon\zeta(x, y, \mathrm{d}y/\mathrm{d}x), \\ y \to \bar{y} &= \bar{y}(x, y, \mathrm{d}y/\mathrm{d}x) = y + \varepsilon\eta(x, y, \mathrm{d}y/\mathrm{d}x), \end{aligned} \tag{4.2}$$

where we include only the first derivative, because the second derivative is defined by (4.1) and can be expressed in terms of the first derivative. Calculating the first extension $(\mathrm{d}y/\mathrm{d}x \equiv y')$,

$$\bar{y}' = \frac{(\partial\bar{y}/\partial x)\,\mathrm{d}x + (\partial\bar{y}/\partial y)\,\mathrm{d}y + (\partial\bar{y}/\partial y')\,\mathrm{d}y'}{(\partial\bar{x}/\partial x)\,\mathrm{d}x + (\partial\bar{x}/\partial y)\,\mathrm{d}y + (\partial\bar{x}/\partial y')\,\mathrm{d}y'},$$

we obtain, to first order in ε,

$$\bar{y}' = y' + \varepsilon\left(\frac{\partial\eta}{\partial x} + \frac{\partial\eta}{\partial y}y' + \frac{\partial\eta}{\partial y'}f - \frac{\partial\zeta}{\partial x}y' - \frac{\partial\zeta}{\partial y}(y')^2 - \frac{\partial\zeta}{\partial y'}y'f\right). \tag{4.3}$$

Now let us write equation (4.1) as a system of two first-order ordinary differential equations (this can always be done):

$$v - y' = 0, \qquad v' - f(x, y, v) = 0; \tag{4.4}$$

we ask for the symmetries of this system†:

$$x \to \bar{x} = x + \varepsilon\zeta(x, y, v), \qquad y \to \bar{y} = y + \varepsilon\eta(x, y, v), \qquad v \to \bar{v} = v + \varepsilon\omega(x, y, v). \tag{4.5}$$

The expression for ω was already calculated in (4.3), and it is necessary only to replace y' by v:

$$\omega = \frac{\partial\eta}{\partial x} + \frac{\partial\eta}{\partial y}v + \frac{\partial\eta}{\partial v}f - \frac{\partial\zeta}{\partial x}v - \frac{\partial\zeta}{\partial y}v^2 - \frac{\partial\zeta}{\partial v}vf. \tag{4.6}$$

This takes account of the first of the pair of equations (4.4), and we need only to

† These symmetries must be considered as local differential and not point transformations, because $v = y'$.

calculate the first extension of v and use the second equation of the pair (4.4):

$$\bar{v}' = \frac{(\partial\bar{v}/\partial x)\,\mathrm{d}x + (\partial\bar{v}/\partial y)\,\mathrm{d}y + (\partial\bar{v}/\partial v)\,\mathrm{d}v}{(\partial\bar{x}/\partial x)\,\mathrm{d}x + (\partial\bar{x}/\partial y)\,\mathrm{d}y + (\partial\bar{x}/\partial v)\,\mathrm{d}v}$$

and to first order in ε

$$\bar{v}' = v' + \varepsilon\left(\frac{\partial\omega}{\partial x} + \frac{\partial\omega}{\partial y}v + \frac{\partial\omega}{\partial v}f - \frac{\partial\zeta}{\partial x}f - \frac{\partial\zeta}{\partial y}vf - \frac{\partial\zeta}{\partial v}f^2\right) = v' + \varepsilon\chi_1. \tag{4.7}$$

Using the second of equations (4.4), we obtain

$$\begin{aligned}\bar{v}' - f(\bar{x}, \bar{y}, \bar{v}) &= v' + \varepsilon\chi_1 - f(x, y, v) - \varepsilon\zeta\frac{\partial f}{\partial x} - \varepsilon\eta\frac{\partial f}{\partial y} - \varepsilon\omega\frac{\partial f}{\partial v}\\ &= v' - f(x, y, v) + \varepsilon\left(\chi_1 - \zeta\frac{\partial f}{\partial x} - \eta\frac{\partial f}{\partial y} - \omega\frac{\partial f}{\partial v}\right) = 0.\end{aligned}$$

So if $v' - f(x, y, v) = 0$, we obtain

$$\chi_1 - \zeta\frac{\partial f}{\partial x} - \eta\frac{\partial f}{\partial y} - \omega\frac{\partial f}{\partial v} = 0,$$

and we end up with two equations, (4.6) and

$$\frac{\partial\omega}{\partial x} + \frac{\partial\omega}{\partial y}v + \frac{\partial\omega}{\partial v}f - \frac{\partial\zeta}{\partial x}f - \frac{\partial\zeta}{\partial y}vf - \frac{\partial\zeta}{\partial v}f^2 - \zeta\frac{\partial f}{\partial x} - \eta\frac{\partial f}{\partial y} - \omega\frac{\partial f}{\partial v} = 0. \tag{4.8}$$

These two equations can be reduced to one second-order partial differential equation with two unknown functions ζ and η and three independent variables (x, y, v):

$$\begin{aligned}&f\left(\frac{\partial\eta}{\partial y} - 2\frac{\partial\zeta}{\partial x}\right) - \zeta\frac{\partial f}{\partial x} - \eta\frac{\partial f}{\partial y} - \frac{\partial\eta}{\partial x}\frac{\partial f}{\partial v} + \frac{\partial^2\eta}{\partial x^2} + v\left[2\frac{\partial^2\eta}{\partial x\partial y} - \frac{\partial^2\zeta}{\partial x^2} - 3\frac{\partial\zeta}{\partial y}f - \left(\frac{\partial\eta}{\partial y} - \frac{\partial\zeta}{\partial x}\right)\frac{\partial f}{\partial v}\right]\\ &\quad + v^2\left(\frac{\partial^2\eta}{\partial y^2} - 2\frac{\partial^2\zeta}{\partial x\partial y} + \frac{\partial\zeta}{\partial y}\frac{\partial f}{\partial v}\right) - \frac{\partial^2\zeta}{\partial y^2}v^3 + 2\frac{\partial^2\eta}{\partial v\partial x}f + \frac{\partial^2\eta}{\partial v^2}f^2 + \frac{\partial\eta}{\partial v}\frac{\partial f}{\partial x}\\ &\quad - 2\frac{\partial\zeta}{\partial v}f^2 + v\left(2\frac{\partial^2\eta}{\partial y\partial v}f + \frac{\partial\eta}{\partial v}\frac{\partial f}{\partial y} - 2\frac{\partial^2\zeta}{\partial v\partial x}f - \frac{\partial^2\zeta}{\partial v^2}f^2 - \frac{\partial\zeta}{\partial v}\frac{\partial f}{\partial x}\right)\\ &\quad - v^2\left(2\frac{\partial^2\zeta}{\partial v\partial y}f + \frac{\partial\zeta}{\partial v}\frac{\partial f}{\partial y}\right) = 0.\end{aligned} \tag{4.9}$$

Here again, because of the overdetermination, we will have an infinite parametric monoid of symmetries, and only in the case of point transformations will we have a group; i.e. if we assume that $\zeta = \zeta(x, y)$, $\eta = \eta(x, y)$,

$$\begin{aligned}&\frac{\partial^2\eta}{\partial x^2} + v\left(2\frac{\partial^2\eta}{\partial x\partial y} - \frac{\partial^2\zeta}{\partial x^2}\right) + v^2\left(\frac{\partial^2\eta}{\partial y^2} - 2\frac{\partial^2\zeta}{\partial x\partial y}\right) - \frac{\partial^2\zeta}{\partial y^2}v^3 + f\left(\frac{\partial\eta}{\partial y} - 2\frac{\partial\zeta}{\partial x}\right) - 3\frac{\partial\zeta}{\partial y}vf\\ &\quad - \zeta\frac{\partial f}{\partial x} - \eta\frac{\partial f}{\partial y} - \frac{\partial\eta}{\partial x}\frac{\partial f}{\partial v} - \left(\frac{\partial\eta}{\partial y} - \frac{\partial\zeta}{\partial x}\right)\frac{\partial f}{\partial v}v + \frac{\partial\zeta}{\partial y}v^2\frac{\partial f}{\partial v} = 0.\end{aligned} \tag{4.10}$$

The differential equation of second order with maximal point symmetry (Cohen 1931, Bluman and Cole 1974) is

$$\mathrm{d}^2y/\mathrm{d}x^2 = 0 \Rightarrow f(x, y, v) = 0. \tag{4.11}$$

So we have to solve the following system of equations:

$$\frac{\partial^2\zeta}{\partial y^2}=0, \qquad 2\frac{\partial^2\eta}{\partial x\partial y}-\frac{\partial^2\zeta}{\partial x^2}=0,$$
$$\frac{\partial^2\eta}{\partial y^2}-2\frac{\partial^2\zeta}{\partial x\partial y}=0, \qquad \frac{\partial^2\eta}{\partial x^2}=0. \tag{4.12}$$

The general solution is

$$\zeta(x, y)=a_1xy+a_2y+c_1x^2+b_1x+b_2,$$
$$\eta(x, y)=c_1xy+c_2x+a_1y^2+d_1y+d_2; \tag{4.13}$$

an eight-parameter group, isomorphic to the projective transformations in the plane.

The knowledge of the point transformation symmetries of a second-order ordinary differential equation can be of help in finding the local differential symmetries of this equation. If we consider the previous example, it can easily be confirmed that

$$\zeta(x, y, v)=a_1(v)xy+a_2(v)y+c_1(v)x^2+b_1(v)x+b_2(v),$$
$$\eta(x, y, v)=c_1(v)xy+c_2(v)x+a_1(v)y^2+d_1(v)y+d_2(v), \tag{4.14}$$

is a solution of (4.9). Equation (4.14) is a local differential symmetry; we only need to replace v by y', and we have an infinite parametric monoid of symmetries, defined by eight arbitrary functions of y'.

Now we can return to the problem of first-order ordinary differential equations; let us take an example to show how we can extract more symmetries:

$$y'=x+y. \tag{4.15}$$

According to (3.2) we will have the following two-parameter group:

$$\zeta=B\left(\frac{x+y+1}{x+y}\right), \qquad \eta=A(x+y+1). \tag{4.16}$$

Differentiating (4.15) again, we will get a second-order ordinary differential equation,

$$y''=1+y', \tag{4.17}$$

whose point transformation symmetries (from (4.10)) form an eight-parameter group:

$$\zeta=(E+F\,\mathrm{e}^{-x})y+B\,\mathrm{e}^{-x}+Ex+Fx\,\mathrm{e}^{-x}+C\,\mathrm{e}^{x}+G,$$
$$\eta=Ey^2+(C\,\mathrm{e}^{x}-F\,\mathrm{e}^{-x}+2Ex+A)y$$
$$+D_1\,\mathrm{e}^{x}+D_2+Cx\,\mathrm{e}^{x}-B\,\mathrm{e}^{-x}-Fx\,\mathrm{e}^{-x}+Ex^2+Ax. \tag{4.18}$$

We now go back to equation (2.14) and introduce (4.18) into it, but taking $f=x+y$ of the first-order ordinary differential equation, we obtain for equation (4.15) the following new six-parameter group:

$$\zeta=(E+B\,\mathrm{e}^{-x})y+B\,\mathrm{e}^{-x}+Ex+Bx\,\mathrm{e}^{-x}+C\,\mathrm{e}^{x}+G,$$
$$\eta=Ey^2+(C\,\mathrm{e}^{x}-B\,\mathrm{e}^{-x}+2Ex+A)y$$
$$+D\,\mathrm{e}^{x}+(A-G)+Cx\,\mathrm{e}^{x}-B\,\mathrm{e}^{-x}-Bx\,\mathrm{e}^{-x}+Ex^2+Ax. \tag{4.19}$$

The following figure summarises this new method of obtaining point transformation symmetries for a first-order non-singular ordinary differential equation:

$$\boxed{y' = f_1(x, y)} \xrightarrow{\text{(differentiation)}} \boxed{y'' = \frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial y} y'}$$

$$\downarrow \left(\text{use (4.10) with } f = \frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial y} v\right)$$

$$\boxed{\xi, \eta} \xrightarrow{\text{(use (2.14) with } f_1)} \boxed{\zeta_1, \eta_1}.$$

The knowledge of the infinitesimal or finite transformation which is obtained via the characteristic equations

$$\frac{\mathrm{d}x}{\zeta(x, y, y')} = \frac{\mathrm{d}y}{\eta(x, y, y')} = \frac{\mathrm{d}y'}{\omega(x, y, y')} = \mathrm{d}\sigma \tag{4.20}$$

again allows us to construct new solutions from known ones for a differential equation of second order, but, what is more important, it helps us to reduce a second-order ordinary differential equation to a first-order ordinary differential equation. For this we have the following theorem, which is a fundamental for similarity analysis.

Theorem (Cohen 1931, Bluman and Cole 1974). If $\mathrm{d}^2y/\mathrm{d}x^2 = f(x, y, y')$ is a differential equation of second order and is conformally invariant under the transformation

$$x \to \bar{x} = x + \varepsilon\zeta(x, y, y'), \qquad y \to \bar{y} = y + \varepsilon\eta(x, y, y'), \qquad y' \to \bar{y}' = y' + \varepsilon\omega(x, y, y'),$$

then the characteristic equations (4.20) produce two conserved quantities $u(x, y, y')$ and $z(x, y, y')$ obtained by integration of (4.20).

Suppose u is a function of z; then second-order ordinary differential equations can be reduced by appropriate substitution into a first-order differential equation†:

$$\mathrm{d}u/\mathrm{d}z = F(u, z).$$

Before passing to an example, let us point out that this theorem can be extended to higher-order ordinary differential equations, which by successive applications can thus be reduced to first order. It has also been extended to partial differential equations or systems (Michal 1952, Morgan 1952), where we can reduce successively the number of variables.

Next it is important to mention the concept of conserved quantity associated with the notion of symmetry as defined by us. The idea that symmetries produce conserved quantities comes from the work of Noether (1918), but there it is necessary to have, in addition to symmetries, also a Lagrangian for the equation in order to obtain conserved quantities. Our method is far superior, because from the equations we obtain the symmetries, and through the characteristic equations the conserved quantities or invariants for the symmetry transformations. Now we understand the importance of obtaining the symmetries of a differential equation; they are the *heart* of the equation.

† The statement of the theorem is more general than that established in the literature, because we include local differential symmetries, but the method of proof is exactly the same.

Example.

$$y'' = 0.$$

This is a very simple example, as are most that we have given. But our intention is to show the method at work, so that it can be applied to very complicated differential equations where all the traditional methods fail and one usually resorts to numerical solutions.

Let us take a particular local differential symmetry of this equation (see (4.14)):

$$\zeta = xy', \qquad \eta = 0.$$

This implies from (4.6) that

$$\omega = -(y')^2,$$

so the characteristic equations are

$$\mathrm{d}x/xy' = \mathrm{d}y/0 = \mathrm{d}y'/-(y')^2.$$

A trivial invariant or conserved quantity is

$$y \to \bar{y} = y,$$

and let us define

$$u(x, y, y') := y.$$

So, we have to deal with the remaining characteristic equation, that must give us the second invariant:

$$\frac{\mathrm{d}x}{xy'} = \frac{\mathrm{d}y'}{-(y')^2} \Rightarrow \frac{\mathrm{d}x}{x} = -\frac{\mathrm{d}y'}{y'},$$

$$\mathrm{d}\ln x = -\mathrm{d}\ln y' + \ln z(u)$$

$$\Rightarrow \ln xy' = \ln z(u),$$

$$z(u) := xy' \leftarrow \text{second invariant}$$

$$\Rightarrow y' = z(u)/x.$$

Introducing this in our equation,

$$y'' = -\frac{z(u)}{x^2} + \frac{1}{x}\frac{\mathrm{d}z}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = 0$$

but

$$\mathrm{d}u/\mathrm{d}x = y' = z(u)/x$$

$$\Rightarrow y'' = -\frac{z(u)}{x^2} + \frac{1}{x^2}\frac{\mathrm{d}z}{\mathrm{d}u}z(u) = 0 \quad \Big| \quad \frac{x^2}{z(u)} \neq 0.$$

We obtain the desired first-order ordinary differential equation

$$\mathrm{d}z(u)/\mathrm{d}u - 1 = 0,$$

whose general solution is

$$z(u) = u - B, \qquad B \text{ is a constant.}$$

But $z(u) = xy'$ and $u = y$, so we have another first-order ordinary differential equation to solve:

$$y' = (y - B)/x$$

$$\Rightarrow dy/(y - B) = \mathrm{d}x/x;$$

we obtain the general solution of the original equation:

$$\ln(y - B) = \ln x + \ln A, \qquad A \text{ is constant,}$$

$$\ln(y - B) = \ln Ax,$$

$$y - B = Ax \Rightarrow y = Ax + B.$$

5. The third-order ordinary differential equation

We now treat the third-order ordinary differential equation, the method being essentially a natural generalisation of what was done previously. The extension to higher-order differential equations is straightforward, and our aim is to show that a non-trivial equation can be dealt with by this method, where otherwise we would have to resort to numerical solutions.

The equation is

$$\mathrm{d}^3y/\mathrm{d}x^3 = f(x, y, \mathrm{d}y/\mathrm{d}x, \mathrm{d}^2y/\mathrm{d}x^2), \tag{5.1}$$

and we can write it as a system of three ordinary differential equations of first order with three dependent variables (y, v, u) and one independent variable x:

$$\mathrm{d}u/\mathrm{d}x - f(x, y, v, u) = 0, \qquad \mathrm{d}v/\mathrm{d}x - u = 0, \qquad \mathrm{d}y/\mathrm{d}x - v = 0. \tag{5.2}$$

Considering local differential transformations for this system,

$$\begin{aligned} x \to \bar{x} &= x + \varepsilon\zeta(x, y, u, v), & y \to \bar{y} &= y + \varepsilon\eta(x, y, u, v), \\ v \to \bar{v} &= v + \varepsilon\omega(x, y, u, v), & u \to \bar{u} &= u + \varepsilon\chi(x, y, u, v), \end{aligned} \tag{5.3}$$

we obtain for the first extensions

$$\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} = \frac{(\partial\bar{y}/\partial x)\,\mathrm{d}x + (\partial\bar{y}/\partial y)\,\mathrm{d}y + (\partial\bar{y}/\partial v)\,\mathrm{d}v + (\partial\bar{y}/\partial u)\,\mathrm{d}u}{(\partial\bar{x}/\partial x)\,\mathrm{d}x + (\partial\bar{x}/\partial y)\,\mathrm{d}y + (\partial\bar{x}/\partial v)\,\mathrm{d}v + (\partial\bar{x}/\partial u)\,\mathrm{d}u}$$

$$= \frac{\mathrm{d}y + \varepsilon[(\partial\eta/\partial x)\,\mathrm{d}x + (\partial\eta/\partial y)\,\mathrm{d}y + (\partial\eta/\partial v)\,\mathrm{d}v + (\partial\eta/\partial u)\,\mathrm{d}u]}{\mathrm{d}x + \varepsilon[(\partial\zeta/\partial x)\,\mathrm{d}x + (\partial\zeta/\partial y)\,\mathrm{d}y + (\partial\zeta/\partial v)\,\mathrm{d}v + (\partial\zeta/\partial u)\,\mathrm{d}u]},$$

$$\begin{aligned} \frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} &= \left[\frac{\mathrm{d}y}{\mathrm{d}x} + \varepsilon\left(\frac{\partial\eta}{\partial x} + \frac{\partial\eta}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial\eta}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\partial\eta}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}x}\right)\right]\left[1 - \varepsilon\left(\frac{\partial\zeta}{\partial x} + \frac{\partial\zeta}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial\zeta}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\partial\zeta}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}x}\right)\right], \\ \frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} &= \frac{\mathrm{d}y}{\mathrm{d}x} + \varepsilon\left(\frac{\partial\eta}{\partial x} + \frac{\partial\eta}{\partial y}v + \frac{\partial\eta}{\partial v}u + \frac{\partial\eta}{\partial u}f - \frac{\partial\zeta}{\partial x}v - \frac{\partial\zeta}{\partial y}v^2 - \frac{\partial\zeta}{\partial v}uv - \frac{\partial\zeta}{\partial u}fv\right), \end{aligned} \tag{5.4}$$

and in similar form

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}\bar{x}} = \frac{\mathrm{d}v}{\mathrm{d}x} + \varepsilon\left(\frac{\partial\omega}{\partial x} + \frac{\partial\omega}{\partial y}v + \frac{\partial\omega}{\partial v}u + \frac{\partial\omega}{\partial u}f - \frac{\partial\zeta}{\partial x}u - \frac{\partial\zeta}{\partial y}vu - \frac{\partial\zeta}{\partial v}u^2 - \frac{\partial\zeta}{\partial u}fu\right), \tag{5.5}$$

$$\frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{x}} = \frac{\mathrm{d}u}{\mathrm{d}x} + \varepsilon\left(\frac{\partial\chi}{\partial x} + \frac{\partial\chi}{\partial y}v + \frac{\partial\chi}{\partial v}u + \frac{\partial\chi}{\partial u}f - \frac{\partial\zeta}{\partial x}f - \frac{\partial\zeta}{\partial y}fv - \frac{\partial\zeta}{\partial v}fu - \frac{\partial\zeta}{\partial u}f^2\right). \tag{5.6}$$

Let us define

$$\frac{d\bar{y}}{d\bar{x}} := \frac{dy}{dx} + \varepsilon\eta_1, \qquad \frac{d\bar{v}}{d\bar{x}} := \frac{dv}{dx} + \varepsilon\omega_1, \qquad \frac{d\bar{u}}{d\bar{x}} := \frac{du}{dx} + \varepsilon\chi_1,$$

and introducing this in equations (5.2) we have to first order in ε

$$\frac{d\bar{u}}{d\bar{x}} - f(\bar{x}, \bar{y}, \bar{v}, \bar{u})$$

$$= \frac{du}{dx} + \varepsilon\chi_1 - f(x, y, v, u) - \varepsilon\left(\zeta\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \omega\frac{\partial f}{\partial v} + \chi\frac{\partial f}{\partial u}\right)$$

$$= \frac{du}{dx} - f(x, y, v, u) + \varepsilon\left(\chi_1 - \zeta\frac{\partial f}{\partial x} - \eta\frac{\partial f}{\partial y} - \omega\frac{\partial f}{\partial v} - \chi\frac{\partial f}{\partial u}\right) = 0,$$

$$\frac{d\bar{v}}{d\bar{x}} - \bar{u} = \frac{dv}{dx} - u + \varepsilon(\omega_1 - \chi) = 0, \qquad \frac{d\bar{y}}{d\bar{x}} - \bar{v} = \frac{dy}{dx} - v + \varepsilon(\eta_1 - \omega) = 0,$$

so that if equation (5.2) is satisfied for a solution, then

$$\chi_1 - \zeta\frac{\partial f}{\partial x} - \eta\frac{\partial f}{\partial y} - \eta_1\frac{\partial f}{\partial v} - \omega_1\frac{\partial f}{\partial u} = 0. \tag{5.7}$$

We would write equation (5.7) as a partial differential equation of third order for ζ and η only, but the expression would be very long and cumbersome. So we shall first consider the point transformation

$$\zeta = \zeta(x, y), \qquad \eta = \eta(x, y). \tag{5.8}$$

Then (5.7) can be written as

$$\frac{\partial\chi}{\partial x} + \frac{\partial\chi}{\partial y}v + \frac{\partial\chi}{\partial v}u + \frac{\partial\chi}{\partial u}f - \frac{\partial\zeta}{\partial x}f - \frac{\partial\zeta}{\partial y}fv - \zeta\frac{\partial f}{\partial x} - \eta\frac{\partial f}{\partial y} - \frac{\partial\eta}{\partial x}\frac{\partial f}{\partial v} - \frac{\partial\eta}{\partial y}v\frac{\partial f}{\partial v} + \frac{\partial\zeta}{\partial x}v\frac{\partial f}{\partial v} + \frac{\partial\zeta}{\partial y}v^2\frac{\partial f}{\partial v}$$

$$-\frac{\partial\omega}{\partial x}\frac{\partial f}{\partial u} - \frac{\partial\omega}{\partial y}v\frac{\partial f}{\partial u} - \frac{\partial\omega}{\partial v}u\frac{\partial f}{\partial u} - \frac{\partial\omega}{\partial u}f\frac{\partial f}{\partial u} + \frac{\partial\zeta}{\partial x}u\frac{\partial f}{\partial u} + \frac{\partial\zeta}{\partial y}vu\frac{\partial f}{\partial u} = 0. \tag{5.9}$$

Equation (5.9) can be written in terms of η and ζ only, if we know that

$$\omega = \eta_1 = \frac{\partial\eta}{\partial x} + \frac{\partial\eta}{\partial y}v - \frac{\partial\zeta}{\partial x}v - \frac{\partial\zeta}{\partial y}v^2$$

and

$$\chi = \omega_1 = \frac{\partial\omega}{\partial x} + \frac{\partial\omega}{\partial y}v + \frac{\partial\omega}{\partial v}u + \frac{\partial\omega}{\partial u}f - \frac{\partial\zeta}{\partial x}u - \frac{\partial\zeta}{\partial y}vu.$$

We obtain

$$\frac{\partial^2\eta}{\partial x^3} + (3\frac{\partial^3\eta}{\partial x^2\partial y} - \frac{\partial^2\zeta}{\partial x^3}\Big)v + \left(3\frac{\partial^3\eta}{\partial x\partial y2} - 3\frac{\partial^3\zeta}{\partial x^2\partial y}\right)v^2 + \left(3\frac{\partial^2\eta}{\partial x\partial y} - 3\frac{\partial^2\zeta}{\partial x^2}\right)u$$

$$+\left(3\frac{\partial^2\eta}{\partial y^2} - 9\frac{\partial^2\zeta}{\partial x\partial y}\right)uv - 3\frac{\partial\zeta}{\partial y}u^2 + \left(\frac{\partial^3\eta}{\partial y^3} - 3\frac{\partial^3\zeta}{\partial x\partial y^2}\right)v^3 - 6\frac{\partial^2\zeta}{\partial y^2}v^2u - \frac{\partial^3\zeta}{\partial y^3}v^4$$

$$+\left(\frac{\partial\eta}{\partial y} - 3\frac{\partial\zeta}{\partial x}\right)f - 4\frac{\partial\zeta}{\partial y}fv - \zeta\frac{\partial f}{\partial x} - \eta\frac{\partial f}{\partial y} - \frac{\partial\eta}{\partial x}\frac{\partial f}{\partial v} - \left(\frac{\partial\eta}{\partial y} - \frac{\partial\zeta}{\partial x}\right)v\frac{\partial f}{\partial v} \tag{5.10}$$

$$+\frac{\partial\zeta}{\partial y}v^2\frac{\partial f}{\partial v}-\frac{\partial^2\eta}{\partial x^2}\frac{\partial f}{\partial u}-\left(2\frac{\partial^2\eta}{\partial x\partial y}-\frac{\partial^2\zeta}{\partial x^2}\right)v\frac{\partial f}{\partial u}-\left(\frac{\partial^2\eta}{\partial y^2}-2\frac{\partial^2\zeta}{\partial x\partial y}\right)v^2\frac{\partial f}{\partial u}$$
$$+\frac{\partial^2\zeta}{\partial y^2}v^3\frac{\partial f}{\partial u}-\left(\frac{\partial\eta}{\partial y}-2\frac{\partial\zeta}{\partial x}\right)u\frac{\partial f}{\partial u}+3\frac{\partial\zeta}{\partial y}uv\frac{\partial f}{\partial u}=0.$$

Example. Taking the third-order differential equation with maximal symmetry, we have

$$\mathrm{d}^3y/\mathrm{d}x^3=0\Rightarrow f\equiv 0,$$

and we obtain the following systems for ζ and η from equation (5.10):

$$\partial^3\eta/\partial x^3=0,\qquad 3\,\partial^3\eta/\partial x^2\partial y-\partial^3\zeta/\partial x^3=0,\qquad \partial^3\eta/\partial x\partial y^2-\partial^3\zeta/\partial x^2\partial y=0,$$
$$3\,\partial^2\eta/\partial x\partial y-3\,\partial^2\zeta/\partial x^2=0,\qquad 3\,\partial^2\eta/\partial y^2-9\,\partial^2\zeta/\partial x\partial y=0,\qquad 3\,\partial\zeta/\partial y=0,$$
$$\partial^3\eta/\partial y^3-3\,\partial^3\zeta/\partial x\partial y^2=0,\qquad 6\,\partial^2\zeta/\partial y^2=0,\qquad \partial^3\zeta/\partial y^3=0,$$

whose general solution is a seven-parameter group:

$$\zeta=A_1x^2+A_2x+A_3,\qquad \eta=(2A_1x+B_2)y+C_1x^2+C_2+C_3.\tag{5.11}$$

So we conclude that any third-order ordinary differential equation can have at most a seven-parameter group of point symmetries.

We now consider local differential symmetries for third-order ordinary differential equations; i.e. we include only first derivatives in ζ and η for the sake of simplicity, the inclusion of second derivatives being straightforward, but cumbersome.

$$\zeta=\zeta(x,y,v),\qquad \eta=\eta(x,y,v).$$

Then (5.7) can be written as

$$\frac{\partial\chi}{\partial x}+\frac{\partial\chi}{\partial y}v+\frac{\partial\chi}{\partial v}u+\frac{\partial\chi}{\partial u}f-\frac{\partial\zeta}{\partial x}f-\frac{\partial\zeta}{\partial y}vf-\frac{\partial\zeta}{\partial v}uf-\zeta\frac{\partial f}{\partial x}-\eta\frac{\partial f}{\partial y}+\frac{\partial\eta}{\partial x}\frac{\partial f}{\partial v}-\frac{\partial\eta}{\partial y}v\frac{\partial f}{\partial v}$$
$$-\frac{\partial\eta}{\partial v}u\frac{\partial f}{\partial v}+\frac{\partial\zeta}{\partial x}v\frac{\partial f}{\partial v}+\frac{\partial\zeta}{\partial y}v^2\frac{\partial f}{\partial v}+\frac{\partial\zeta}{\partial v}uv\frac{\partial f}{\partial v}-\frac{\partial\omega}{\partial x}\frac{\partial f}{\partial u}-\frac{\partial\omega}{\partial y}v\frac{\partial f}{\partial u}-\frac{\partial\omega}{\partial v}u\frac{\partial f}{\partial u}$$
$$-\frac{\partial\omega}{\partial u}f\frac{\partial f}{\partial u}+\frac{\partial\zeta}{\partial x}u\frac{\partial f}{\partial u}+\frac{\partial\zeta}{\partial y}vu\frac{\partial f}{\partial u}+\frac{\partial\zeta}{\partial v}u^2\frac{\partial f}{\partial u}=0.\tag{5.12}$$

Writing (5.12) in terms of η and ζ only, if we know that

$$\omega=\eta_1=\frac{\partial\eta}{\partial x}+\frac{\partial\eta}{\partial y}v+\frac{\partial\eta}{\partial v}u-\frac{\partial\zeta}{\partial x}v-\frac{\partial\zeta}{\partial y}v^2-\frac{\partial\zeta}{\partial v}uv$$

and

$$\chi=\omega_1=\frac{\partial\omega}{\partial x}+\frac{\partial\omega}{\partial y}v+\frac{\partial\omega}{\partial v}u+\frac{\partial\omega}{\partial u}f-\frac{\partial\zeta}{\partial x}u-\frac{\partial\zeta}{\partial y}vu-\frac{\partial\zeta}{\partial v}u^2,$$

we obtain

$$\frac{\partial^3\eta}{\partial x^3}+\left(3\frac{\partial^3\eta}{\partial x^2\partial y}-\frac{\partial^3\zeta}{\partial x^3}\right)v+\left(3\frac{\partial^3\eta}{\partial y^2\partial x}-3\frac{\partial^3\zeta}{\partial x^2\partial y}\right)v^2+\left(\frac{\partial^3\eta}{\partial y^3}-3\frac{\partial^3\zeta}{\partial y^2\partial x}\right)v^3-\frac{\partial^3\zeta}{\partial y^3}v^4$$
$$+\left(3\frac{\partial^3\eta}{\partial x^2\partial v}+3\frac{\partial^2\eta}{\partial x\,\partial y}-3\frac{\partial^2\zeta}{\partial x^2}\right)u+\left(3\frac{\partial^2\eta}{\partial v^2\partial x}-6\frac{\partial^2\zeta}{\partial v\,\partial x}+3\frac{\partial^2\eta}{\partial y\,\partial v}-3\frac{\partial\zeta}{\partial y}\right)u^2$$

$$+\left(\frac{\partial^3\eta}{\partial v^3}-3\frac{\partial^2\zeta}{\partial v^2}\right)u^3+\left(6\frac{\partial^3\eta}{\partial x\,\partial y\,\partial v}-3\frac{\partial^3\zeta}{\partial x^2\,\partial v}-9\frac{\partial^2\zeta}{\partial x\,\partial y}+3\frac{\partial^2\eta}{\partial y^2}\right)uv$$

$$+\left(3\frac{\partial^3\eta}{\partial y^2\,\partial v}-6\frac{\partial^3\zeta}{\partial x\,\partial y\,\partial v}-6\frac{\partial^2\zeta}{\partial y^2}\right)uv^2+\left(3\frac{\partial^3\eta}{\partial v^2\,\partial y}-3\frac{\partial^3\zeta}{\partial v^2\,\partial x}-9\frac{\partial^2\zeta}{\partial y\,\partial v}\right)u^2v$$

$$-3\frac{\partial^3\zeta}{\partial v^2\,\partial y}u^2v^2-3\frac{\partial^3\zeta}{\partial v\,\partial y^2}v^3u-\frac{\partial^3\zeta}{\partial v^3}vu^3+\left(3\frac{\partial^2\eta}{\partial v\,\partial x}+\frac{\partial\eta}{\partial y}-3\frac{\partial\zeta}{\partial x}\right)f$$

$$+\left(3\frac{\partial^2\eta}{\partial v\,\partial y}-3\frac{\partial^2\zeta}{\partial v\,\partial x}-4\frac{\partial\zeta}{\partial y}\right)vf+\left(3\frac{\partial^2\eta}{\partial v^2}-6\frac{\partial\zeta}{\partial v}\right)uf+\left(\frac{\partial\eta}{\partial v}-\zeta\right)\frac{\partial f}{\partial x}$$

$$-\frac{\partial\zeta}{\partial v}v\frac{\partial f}{\partial x}+\frac{\partial\eta}{\partial v}v\frac{\partial f}{\partial y}-3\frac{\partial^2\zeta}{\partial v\,\partial y}v^2f-\frac{\partial\zeta}{\partial v}v^2\frac{\partial f}{\partial y}-3\frac{\partial^2\zeta}{\partial v^2}vuf-\eta\frac{\partial f}{\partial y}-\frac{\partial\eta}{\partial x}\frac{\partial f}{\partial v}$$

$$+\left(\frac{\partial\zeta}{\partial x}-\frac{\partial\eta}{\partial y}\right)v\frac{\partial f}{\partial v}+\frac{\partial\zeta}{\partial y}v^2\frac{\partial f}{\partial v}+\left(2\frac{\partial\zeta}{\partial x}-2\frac{\partial^2\eta}{\partial v\,\partial x}-\frac{\partial\eta}{\partial y}\right)u\frac{\partial f}{\partial u}$$

$$+\left(3\frac{\partial\zeta}{\partial y}+2\frac{\partial^2\zeta}{\partial v\,\partial x}-2\frac{\partial^2\eta}{\partial v\,\partial y}\right)vu\frac{\partial f}{\partial u}+\left(2\frac{\partial\zeta}{\partial v}-\frac{\partial^2\eta}{\partial v^2}\right)u^2\frac{\partial f}{\partial u}-\frac{\partial^2\eta}{\partial x^2}\frac{\partial f}{\partial u}$$

$$+\left(\frac{\partial^2\zeta}{\partial x^2}-2\frac{\partial^2\eta}{\partial y\,\partial x}\right)v\frac{\partial f}{\partial u}+\left(2\frac{\partial^2\eta}{\partial x\,\partial y}-\frac{\partial^2\eta}{\partial y^2}\right)v^2\frac{\partial f}{\partial u}+\frac{\partial^2\zeta}{\partial y^2}v^3\frac{\partial f}{\partial u}$$

$$+2\frac{\partial^2\zeta}{\partial v\,\partial y}uv^2\frac{\partial f}{\partial u}+\frac{\partial^2\zeta}{\partial v^2}u^2v\frac{\partial f}{\partial u}=0. \qquad (5.13)$$

Again, the knowledge of the point transformation of our equation

$$\mathrm{d}^3y/\mathrm{d}x^3=0$$

can be helpful in obtaining a local differential symmetry for this equation. From (5.11), we have the ansatz

$$\zeta=A_1(v)x^2+A_2(v)x+A_3(v),$$
$$\eta=2A_1(v)xy+B_2(v)y+C_1(v)x^2+C_2(v)x+C_3(v),$$

and introducing this in (5.13), we obtain as a general solution for $f\equiv 0$

$$\zeta=Ax+B(v), \qquad \eta=Cy+Dx^2+Evx+Fx+vB(v)+Gv^2+H,$$

where A, C, D, E, F, G and H are constants.

6. Non-trivial examples

All the previous examples of differential equations have been very simple, since our aim was didactical. Now we want to deal with a non-trivial example to show the strength of the similarity analysis method.

The Thomas–Fermi differential equation (1926–1928) arises from a statistical model of a many-electron atom:

$$\mathrm{d}^2y/\mathrm{d}x^2-y^{3/2}/x^{1/2}=0. \qquad (6.1)$$

We begin by looking for the point transformation symmetry of equation (6.1):

$$y''=x^{-1/2}y^{3/2}\Rightarrow f(x,y,v)=x^{-1/2}y^{3/2}. \tag{6.2}$$

Introducing equation (6.2) into equation (4.10),

$$\eta_{xx}+v(2\eta_{xy}-\zeta_{xx})+v^2(\eta_{yy}-2\zeta_{xy})-v^3\zeta_{yy}+x^{-1/2}y^{3/2}(\eta_y-2\zeta_x) \\ -v\cdot 3x^{-1/2}y^{3/2}\zeta_y+\tfrac{1}{2}x^{-3/2}y^{3/2}\zeta-\tfrac{3}{2}x^{-1/2}y^{1/2}\eta=0. \tag{6.3}$$

Grouping in powers of v, we obtain four differential equations for ζ and η:

$$\zeta_{yy}=0, \tag{6.4}$$

$$\eta_{yy}-2\zeta_{xy}=0, \tag{6.5}$$

$$2\eta_{xy}-\zeta_{xx}-3x^{-1/2}y^{3/2}\zeta_y=0, \tag{6.6}$$

$$\eta_{xx}+x^{-1/2}y^{3/2}(\eta_y-2\zeta_x)+\tfrac{1}{2}x^{-3/2}y^{3/2}\zeta-\tfrac{3}{2}x^{-1/2}y^{1/2}\eta=0. \tag{6.7}$$

A straightforward calculation gives the general solution of this system:

$$\zeta(x)=-\tfrac{1}{3}kx, \qquad \eta(y)=ky. \tag{6.8}$$

So we have the following infinitesimal point transformation symmetry for equation (6.1):

$$x\to\bar{x}=x-\varepsilon\frac{k}{3}x, \qquad y\to\bar{y}=y+\varepsilon ky \qquad (k \text{ is an arbitrary real constant}). \tag{6.9}$$

The infinitesimal transformations (6.9) define a one-parameter Lie group, and the finite transformations are

$$\bar{x}=x\,\mathrm{e}^{-k\sigma/3}, \qquad \bar{y}=y\,\mathrm{e}^{k\sigma}. \tag{6.10}$$

Writing down the characteristic equations,

$$\mathrm{d}x/-\tfrac{1}{3}x=\mathrm{d}y/y=\mathrm{d}y'/\tfrac{4}{3}y', \tag{6.11}$$

the corresponding invariants are

$$u=yx^3, \qquad z(u)=x^4y'\Rightarrow y'=z(u)/x^4. \tag{6.12}$$

Introducing equation (6.12) into equation (6.1), we reduce it to a first-order ordinary differential equation for $z(u)$:

$$\frac{\mathrm{d}z(u)}{\mathrm{d}u}=\frac{4z(u)+u^{3/2}}{z(u)+3u}. \tag{6.13}$$

An exceptional solution of equation (6.1) can be found with the help of equation (6.13), knowing that its singular point is

$$u_s=144, \qquad z_s=-432. \tag{6.14}$$

The singular solution of (6.1) is then

$$y_s(x)=144/x^3. \tag{6.15}$$

Now we consider local differential symmetry transformations for equation (6.1) and make the ansatz

$$\zeta=0, \qquad \eta(x,y,v)=\sum_{n=0}^{R}a_n(x,y)v^n, \qquad R\in\mathbb{N}. \tag{6.16}$$

Introducing equation (6.16) into equation (4.9), we obtain after a straightforward but long calculation

$$\zeta = 0, \qquad \eta = \tfrac{1}{3}Mxv + My \qquad (M \text{ is an arbitrary real constant}). \tag{6.17}$$

From (6.17) we can calculate ω (see equation (4.6)), and we obtain

$$\omega = \tfrac{4}{3}My' + \tfrac{1}{3}Mx^{1/2}y^{3/2}. \tag{6.18}$$

As we said previously, the knowledge of a particular solution and of the infinitesimal transformation allows us to construct a new solution. Equation (2.12) is valid for point transformations and not for local differential symmetries, but the generalisation is straightforward:

$$\begin{aligned} \bar{x} &= x + \varepsilon\zeta(x, y, v) + (\varepsilon^2/2!)(\zeta\zeta_x + \eta\zeta_y + \omega\zeta_v) + \ldots, \\ \bar{y} &= y + \varepsilon\eta(x, y, v) + (\varepsilon^2/2!)(\zeta\eta_x + \eta\eta_y + \omega\eta_v) + \ldots. \end{aligned} \tag{6.19}$$

Inserting (6.17) and (6.18) into (6.19), we obtain a new solution in terms of a known one:

$$\bar{y} = y + \bar{\varepsilon}\left(\frac{x}{3}\frac{\mathrm{d}y}{\mathrm{d}x} + y\right) + \frac{\bar{\varepsilon}^2}{2!}\left(y + \frac{7}{9}x\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{9}x^{3/2}y^{3/2}\right) + \ldots, \tag{6.20}$$

where we have absorbed the constant M in the infinitesimal $\bar{\varepsilon}$,

$$\bar{\varepsilon} := \varepsilon M.$$

It is simple to show that the exceptional solution equation (6.15) is a similarity solution and does not produce new solutions; it is enough to introduce (6.15) into (6.20) and we obtain $\bar{y} = y$.

But we hope there are other known solutions which do give us new results.

(a) *Sommerfeld solution.*

$$y(x) = \frac{1}{x^3}\left(144 + \frac{C}{x^{(1-\sqrt{73})/2}} + \ldots\right) \qquad \text{valid for } x \to \infty,\ C \text{ an arbitrary constant.} \tag{6.21}$$

(b) The theorem of existence and uniqueness valid for ordinary differential equations allows us to build a solution which is analytic at a point $x_0 \neq 0$ and converges in the neighbourhood of x_0:

$$\begin{aligned} y(x) = A &+ (1/2!)x_0^{-1/2}A^{3/2}(x - x_0)^2 - (1/3!2)x_0^{-3/2}A^{3/2}(x - x_0)^3 \\ &+ (1/4!)(\tfrac{3}{4}x_0^{-5/2}A^{3/2} + \tfrac{3}{2}x_0^{-1}A^2)(x - x_0)^4 \\ &- (1/5!)(\tfrac{15}{8}x_0^{-7/2}A^{3/2} + 3x_0^{-2}A^2)(x - x_0)^5 + \ldots, \end{aligned} \tag{6.22}$$

where $y(x_0) = A \neq 0$ and $y'(x_0) = 0$.

(c) *Fermi solution.*

$$y(x) = 1 + cx + \tfrac{4}{3}x^{3/2} + \tfrac{6}{15}cx^{5/2} + \ldots \qquad \text{valid in the neighbourhood of } x = 0. \tag{6.23}$$

Introducing solutions ((6.21), (6.22) and (6.23)) into equation (6.20), we obtain new solutions of (6.1) for any x.

Another ordinary differential equation of interest in boundary layer problems is the Blasius equation

$$d^3y/dx^3 = y\, d^2y/dx^2, \tag{6.24}$$

where we know two solutions in closed form

$$y_1(x) = ax + b, \qquad y_2(x) = -3/(x+c); \qquad a, b, c \text{ are arbitrary constants.} \tag{6.25}$$

Under the ansatz

$$\zeta \equiv 0, \qquad \eta = \eta(x, y, v), \tag{6.26}$$

and introducing it into equations (5.13), we obtain as general solution for the infinitesimal η

$$\eta(x, y, v) = Axv + Bv + Ay. \tag{6.27}$$

New solutions can be built from equation (6.19) after introducing (6.25):

$$\bar{y} = y + \bar{\varepsilon}\left((x+D)\frac{dy}{dx} + y\right) + \frac{\bar{\varepsilon}^2}{2}\left(3(x+D)\frac{dy}{dx} + (x+D)^2\frac{d^2y}{dx^2} + y\right) + \ldots \tag{6.28}$$

where $\bar{\varepsilon} := \varepsilon A$ and $D := B/A$.

7. Conclusion

There are still some open questions concerning this new method of using local differential symmetries of a differential equation to reduce the order of the equation or to generate new solutions from known ones. The question of initial, boundary or mixed value problems will be treated in a forthcoming publication, as well as how to deal with partial differential equations or systems of them.

The relation between the method of Bäcklund transformations and ours is only in the fact that both involve derivatives. The Bäcklund transformations, however, form neither a group nor a monoid, and there is no canonical form to obtain them, which makes our method superior.

Under Bäcklund transformations we understand the symmetries of a differential equation that have the form

$$v \to \bar{v} = \bar{v}(x, y, \bar{y}, v), \tag{7.1}$$

i.e. the transformed unknown function $\bar{y}(x)$ appears explicitly in the transformation of v, whereas in our method it does not appear. Bäcklund transformations are related to the existence of soliton solutions of a differential equation, and we will show in a forthcoming publication that our method will produce them also, but in a canonical and straightforward manner.

Finally, we can ask if every ordinary differential equation can be solved by our method, thus avoiding the computer completely. The answer is no, and we give the following example:

$$dy/dx = y^2 + x^2. \tag{7.2}$$

This equation, as the reader can show for himself, does not produce solvable differential equations for ζ and η, and it must be solved by numerical methods. The fact that the differential equations for the infinitesimals ζ and η cannot be solved is the signal that only numerical methods can be used. On the other hand, it should be pointed out that the computer or numerical methods do not solve every differential equation; questions of asymptotic behaviour are very difficult or practically impossible to handle with the computer, but relatively simple by our method. In no way do we want to replace the computer by our method (nor can we), but we would like to emphasise that it is worthwhile to try it first, before resorting to the computer. An exact solution, or an expansion in terms of a parameter, is better than numerical tables!

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